

# Agent Arrangement Problem

Tomoki Nakamigawa<sup>1</sup>

Department of Information Science  
Shonan Institute of Technology  
1-1-25 Tsujido-Nishikaigan, Fujisawa 251-8511, Japan

Tadashi Sakuma<sup>2</sup>

Systems Science and Information Studies  
Faculty of Education, Art and Science  
Yamagata University  
1-4-12 Kojirakawa, Yamagata 990-8560, Japan

## Abstract

An *arrangement* of an ordered pair  $(G_A, G_M)$  of graphs is defined as a function  $f$  from  $V(G_A)$  to  $V(G_M)$  such that, for each vertex  $c$  of  $G_M$ , the vertex-set  $f^{-1}(c)$  of  $G_A$  either is  $\emptyset$  (the case when  $c \notin f(V(G_A))$ ) or induces a connected subgraph of  $G_A$  and that the family  $\{f^{-1}(y) : y \in V(G_M), f^{-1}(y) \neq \emptyset\}$  is a partition of  $V(G_A)$ . Let  $f$  be an arrangement of  $(G_A, G_M)$ , let  $pq$  be an edge of  $G_M$  and let  $U$  be a subset of  $f^{-1}(p)$  such that each of the three graphs  $G_A[U]$ ,  $G_A[f^{-1}(p) \setminus U]$  and  $G_A[f^{-1}(q) \cup U]$  is either connected or  $\emptyset$  and that  $(f^{-1}(p) \cup f^{-1}(q)) \setminus U \neq \emptyset$ . A *transfer* of  $U$  from  $p$  to  $q$  is defined as the modification  $f'$  of  $f$  such that  $f'(x) := f(x)$  for every  $x \notin U$  and  $f'(u) := q$  for every  $u \in U$ . Two arrangements  $f$  and  $g$  of  $(G_A, G_M)$  are called *t-equivalent* if they can be transformed into each other by a finite sequence of transfers. An ordered pair  $(G_A, G_M)$  of graphs is called *almighty* if every two arrangements of the pair  $(G_A, G_M)$  are t-equivalent. In this study, we consider the following two decision problems.

- (P1)** For a given pair of arrangements  $f$  and  $g$  of a given ordered pair  $(G_A, G_M)$  of graphs, decide whether  $f$  is t-equivalent to  $g$  or not.

---

<sup>1</sup>Email: [nakami@info.shonan-it.ac.jp](mailto:nakami@info.shonan-it.ac.jp)

<sup>2</sup>This work was supported by Grant-in-Aid for Scientific Research (C).

Email: [sakuma@e.yamagata-u.ac.jp](mailto:sakuma@e.yamagata-u.ac.jp)

**(P2)** For a given ordered pair  $(G_A, G_M)$  of graphs, decide whether the pair  $(G_A, G_M)$  is almighty or not.

We show an  $O(|E(G_A)| + (|V(G_M)| + |E(G_A)|)|V(G_A)|)$ -time algorithm for **(P1)**. By using this algorithm, we can also construct an explicit sequence of transfers from  $f$  to  $g$  of  $\Theta(|V(G_M)|^2 \cdot |V(G_A)|)$ -length. Lastly we prove the co- $\mathcal{NP}$ -completeness of **(P2)**.

keywords: pebble motion, motion planning

## 1 Introduction

In this paper, we introduce two generalizations of the pebble motion problems [3, 4, 5, 6, 9], the first of which we call “the Subgraph Allocation Problem”. The second is a specialization of the first, which we call “the Agent Arrangement Problem”. We take up this specialization because it has some significance as a theoretical model of logistics.

Throughout this paper, a graph is undirected with no loop or multiple edge. For a graph  $G$ , let  $V(G)$  and  $E(G)$  denote the vertex set of  $G$  and the edge set of  $G$ . Let  $G_A$  be a simple undirected graph such that its vertex set  $V(G_A)$  is the set of *agents* and that  $G_A$  has an edge  $uv$  if and only if the two agents  $u$  and  $v$  have a method of taking mutual communication. Let us call the graph  $G_A$  an *agent network*. On the other hand, let  $G_M$  be a simple undirected graph such that its vertex set  $V(G_M)$  is the set of *countries* and that  $G_M$  has an edge  $pq$  if and only if there exists a way of mutually direct transportation between the two countries  $p$  and  $q$ . Let us call the graph  $G_M$  a *route map*. An *arrangement* of the ordered pair  $(G_A, G_M)$  of graphs is defined as a function  $f$  from  $V(G_A)$  to  $V(G_M)$  such that the family of sets  $\{f^{-1}(c) : c \in V(G_M), f^{-1}(c) \neq \emptyset\}$  is a partition of  $V(G_A)$  and each set  $f^{-1}(c)$  in the family (i.e. the agents staying in the country  $c$ ) induces a connected subgraph of  $G_A$ . Let  $f$  be an arrangement of  $(G_A, G_M)$ , let  $st$  be an edge of  $G_M$  and let  $U(\subseteq f^{-1}(t))$  be a set of agents staying in the country  $s$  such that each of the three graphs  $G_A[U]$ ,  $G_A[f^{-1}(s) \setminus U]$  and  $G_A[f^{-1}(t) \cup U]$  is either connected or  $\emptyset$ . A *transfer* of the set of agents  $U$  along with the edge  $st$  of  $G_M$  from the country  $s$  to the country  $t$  is defined as the modification  $f'$  of  $f$  such that  $f'(x) := f(x)$  for every  $x \notin U$  and  $f'(u) := t$  for every  $u \in U$ . Note that the modification  $f'$  is again an arrangement of the pair  $(G_A, G_M)$ . Here we use the *Subgraph Allocation Problem* (*SGA*, for short) as

a general term for the set of related problems dealing with arrangements and agent-transfers on given ordered pairs of graphs.

In addition to the previous definition of a *transfer* of a set of agents  $U$  from a country  $s$  to a country  $t$ , we sometimes need to assume that such a transfer is allowed if and only if at least one of the following two conditions hold true:

1.  $f^{-1}(t) \neq \emptyset$ .
2.  $f^{-1}(s) \setminus U \neq \emptyset$ .

The meaning of these two additional conditions can be interpreted as follows. The first condition describes the situation when the agent subnetwork  $G_A[U]$  will move from the country  $s$  to the country  $t$  relying on the connection to the agent subnetwork  $G_A[f^{-1}(t)]$  in the country  $t$ . If  $f^{-1}(t) = \emptyset$  and the ‘vanguard’ agent subnetwork  $G_A[U]$  of the agent organization  $G_A$  will pioneer the new territory  $t$ , then  $G_A[U]$  must need backup support of the remainder agent subnetwork  $G_A[f^{-1}(s) \setminus U]$  in the home ground  $s$ . Hence the second condition is necessary if the first condition does not hold. Note that the above pair of conditions can be summarized as the single condition  $f^{-1}(\{s, t\}) \setminus U \neq \emptyset$ . When we impose this restriction on a transfer, to avoid the confusion to the unrestricted version, we use the term the *Agent Arrangement Problem* (AAP, for short) instead of SGA. The Agent Arrangement Problem can be regarded as a natural model for logistics, that is, an appropriate treatment for (re-)configurations on (distribution and/or human) networks in security. And hence it has clear applications to the wide area such as computer science, engineering, social and political science.

Note that both of the SGA and AAP models are generalizations of the pebble motion problem. Actually, on the SGA model, if we assume the agent network  $G_A$  to be an edge-less graph, the set of agents  $V(G_A)$  can be regarded as the set of pebbles on the board graph  $G_M$ . In this case, a transfer of an agent turns to be a move of a pebble on the board. In the same way, on the AAP model, if we assume the agent network  $G_A$  to be a disjoint union of two-vertex complete graphs, and if we exclude from the consideration the meaningless arrangements  $f$  which contain a ‘frozen’ pair  $\{u, v\}$  of agents (that is, an edge  $\{u, v\}$  of  $G_A$  such that  $\{f(u), f(v)\} \notin E(G_M)$ ), then the behavior of the model is essentially the same as the behavior of the pebble motion problem. Of course, for general cases, the behavior of transfers on

SGA or AAP is far from the behavior of pebble motions, and its analysis is considerably difficult issue as is shown in Section 6.

On both of the SGA and AAP models, two arrangements  $f$  and  $g$  of an ordered pair  $(G_A, G_M)$  of graphs is called *t-equivalent* and is denoted by  $f \cong g$  if the two arrangements can be transformed into each other by a finite sequence of transfers. An ordered pair  $(G_A, G_M)$  of graphs is called *almighty* if every two arrangements of the pair  $(G_A, G_M)$  are t-equivalent. In this study, we consider the following two decision problems.

- (P1) For a given two arrangements  $f$  and  $g$  of a given ordered pair  $(G_A, G_M)$  of graphs, decide whether  $f$  is t-equivalent to  $g$  or not.
- (P2) For a given ordered pair  $(G_A, G_M)$  of graphs, decide whether the pair  $(G_A, G_M)$  is almighty or not.

First, by using the previous results of Wilson [9] and Kornhauser et al. [5], we will give a proper description of several good characterizations to the equivalence decision (and some related problems) for the classical pebble motion problem, which we need in our polynomial algorithm for (P1). We provide these in Sections 3 and 4. In Section 5, we describe an  $O(|E(G_M)| + (|V(G_M)| + |E(G_A)|)|V(G_A)|)$ -time algorithm for (P1) on AAP and SGA, in common. By using this algorithm, we can also construct an explicit sequence of transfers from  $f$  to  $g$  of  $\Theta(|V(G_M)|^2 \cdot |V(G_A)|)$ -length. In Section 6, we prove the co- $\mathcal{NP}$ -completeness of (P2) for the both of SGA and AAP.

## 2 Preliminary for the Pebble Motion Problem

Let  $G$  be a connected graph with  $n$  vertices. Let  $P$  be the set of labeled pebbles of order  $m < n$ . A *configuration* of the set of pebbles  $P$  on  $G$  is defined as an injective function  $f$  from  $P$  to  $V(G)$ , where if  $f^{-1}(v) \neq \emptyset$ , then  $f^{-1}(v)$  represents a pebble of  $P$  on the vertex  $v$  of  $G$ , and if  $f^{-1}(v) = \emptyset$ , it means that  $v$  is unoccupied. Any pebble  $p \in P$  must be on some vertex of the graph. Hence we have  $|f^{-1}(v)| \leq 1$  for any  $v \in V(G)$ .

A *move* is transferring a pebble to an adjacent unoccupied vertex. For a pair of configurations  $f$  and  $g$ , we say that  $f$  and  $g$  are *equivalent* if  $f$  can

be transformed into  $g$  by a sequence of finite moves. we write  $f \sim g$  if  $f$  and  $g$  are equivalent.

Let us define the puzzle graph  $\text{puz}(G, k)$  of a graph  $G$  with  $k$  unoccupied vertices such that  $V(\text{puz}(G, k))$  is the set of all the configurations  $\mathcal{F}(G)$ , and  $E(\text{puz}(G, k)) = \{(f, g) : f, g \in \mathcal{F}(G), f \text{ can be transformed into } g \text{ by a single move}\}$ . For example, if  $G$  is a  $4 \times 4$  grid graph,  $\text{puz}(G, 1)$  corresponds to a well-known “15 puzzle” [1, 4, 8].

We say that  $(G, k)$  is *transitive* if for any configuration  $f$  and for any pebble  $p$  of  $P$ ,  $p$  can be shifted to an arbitrary vertex of  $G$  by a sequence of finite moves. For a graph  $G$ , let  $c(G)$  be the number of connected components of  $G$ . We say that  $(G, k)$  is *feasible* if  $c(\text{puz}(G, k)) = 1$ .

Wilson studied the problem for the case  $k = 1$  [9]. It is not difficult to see that  $(G, 1)$  is transitive if and only if  $G$  is 2-connected. Let  $S_m$  and  $A_m$  denote the symmetric group and the alternating group of order  $m$ , respectively. For a finite set  $M$ , let  $S(M)$  be the symmetric group on  $M$ . For a vertex  $x$  of  $V(G)$ , let  $\mathcal{F}_x$  be the set of configurations  $f$  with  $f^{-1}(x) = \emptyset$  and define  $G_x$  as the set of permutations  $\sigma \in S(V(G))$  such that  $\sigma(x) = x$  and for any  $f \in \mathcal{F}_x$ ,  $f$  can be transformed into  $\sigma \circ f$  by a sequence of finite moves. Then (1)  $G_x$  is isomorphic to a subgroup of  $S_{n-1}$ , (2)  $G_x$  is independent on  $x$  up to isomorphism, and (3)  $c(\text{puz}(G, 1)) = [S_{n-1} : G_x] = (n-1)!/|G_x|$ .

For positive integers  $a_1, a_2, a_3$ , we define  $\theta(a_1, a_2, a_3)$ -graph such that (1) there exists a pair of vertices  $u$  and  $v$  of degree 3, and (2)  $u$  and  $v$  are linked by three disjoint paths containing  $a_1, a_2$  and  $a_3$  inner vertices, respectively.

**Theorem A (Wilson[9]).** Let  $n \geq 2$ . Let  $G$  be a graph with  $n$  vertices. Suppose that  $G$  is 2-connected and  $G$  is not a cycle. Let  $c = c(\text{puz}(G, 1))$ .

- (1) If  $G$  is a bipartite graph, then  $G_x \cong A_{n-1}$  and  $c = 2$ .
- (2) If  $G$  is not a bipartite graph except  $\theta(1, 2, 2)$ , then  $G_x \cong S_{n-1}$  and  $c = 1$ .
- (3) If  $G$  is  $\theta(1, 2, 2)$ , then  $G_x \cong PGL_2(5)$  and  $c = 6$ , where  $PGL_2(5)$  is the projective general linear group on 2-dimensional vector space over a finite field of order 5.

Theorem A is generalized for the case  $q \geq 2$  [5]. Let  $G$  be a connected graph. Let  $k$  be a positive integer. A path  $I = v_1 v_2 \dots v_k$  of  $G$  is called an *isthmus* if  $V(G) \setminus V(I)$  is partitioned into nonempty partite sets  $X$  and  $Y$  such that every path from  $X$  to  $Y$  passes through  $I$ . In this definition, we say that the isthmus  $I$  *separates*  $X$  and  $Y$ . An isthmus with  $k$  vertices is called a  $k$ -isthmus. Note that a 1-isthmus is a cut-vertex of  $G$ .

**Theorem B(Kornhauser, Miller, Spirakis[5]).** Let  $2 \leq k \leq n - 1$ . Let  $G$  be a connected graph with  $n$  vertices. Suppose that  $G$  is not a cycle. Then the following conditions are equivalent.

- (1)  $G$  has no  $k$ -isthmus.
- (2)  $(G, k)$  is transitive.
- (3)  $(G, k)$  is feasible.

In application, it is important to consider the number of moves that are necessary, or algorithms to transfer the pebbles. Motion planning on graphs are studied intensively[2, 3, 6, 7].

In the next two sections, we will focus on analyzing the following three problems for pebble motion on graphs, each of which play a key role in the latter sections.

- (1) *transitivity problem*: What is the reachable set of vertices for a given pebble?
- (2) *contact problem*: Can a given pair of pebbles contact each other?
- (3) *equivalence problem*: Can a given pair of configurations be equivalent to each other?

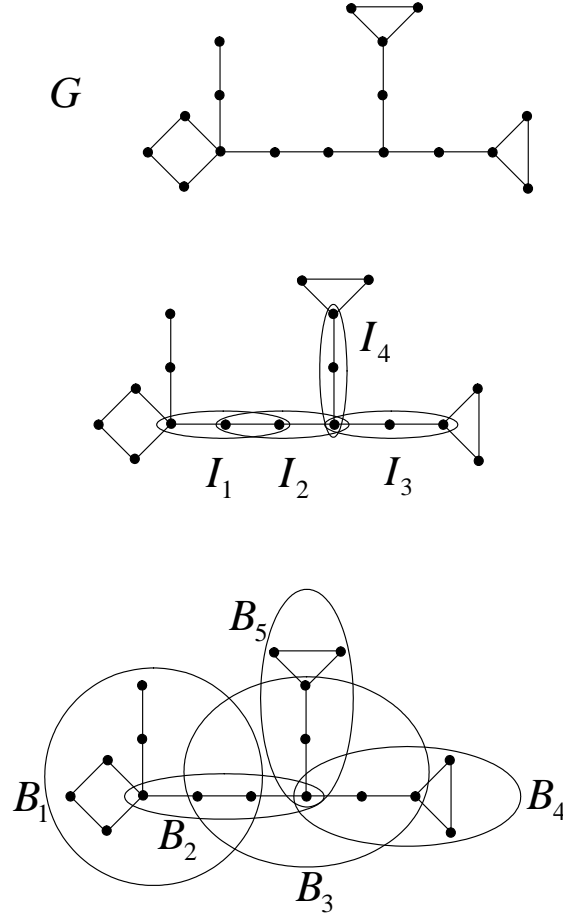
In Section 3, we investigate isthmus structure of graphs and introduce the  $(k)$ -isthmus tree of an underlying graph. The isthmus tree describes how the underlying graph contains its isthmuses. In Section 4, we will see that the equivalence problem (3) for a given pair of configurations can be reduced to the contact problem (2) for each of the configurations. In order to solve these problems, it turns to be useful to use the notion of  $k$ -isthmus tree, where  $k$  is the number of unoccupied vertices of the board graph. Note that an algorithm which solves the equivalence problem (2) in the above and generates an efficient sequence of moves from one configuration to the other, if any, in  $O(|V(G)|^3)$ -time, was already announced by Kornhauser et al. [5]. However, they have never presented its details.

### 3 Isthmus Structure of Graphs

Let  $G$  be a connected simple graph. Let us denote a subgraph of  $G$  induced by  $S \subset V(G)$  by  $G[S]$ . A set of vertices  $B$  of  $G$  is called a  $k$ -block if (1)  $G[B]$  is connected, (2)  $G[B]$  has no  $k$ -isthmus of  $G[B]$ , and (3)  $B$  is maximal with respect to (1) and (2). Namely, for any proper superset  $S$  of  $B$ ,  $G[S]$  is disconnected or  $G[S]$  has a  $k$ -isthmus of  $G[S]$ .

Note that a 1-block is simply a block of a given graph.

We have  $|B| \geq k + 1$  for any  $k$ -block  $B$  of  $G$  with  $B \neq V(G)$ , because  $G[C]$  has no  $k$ -isthmus for  $C \subset V(G)$  with  $|C| \leq k + 1$ . For example, the graph  $G$  in Fig.1 has four 3-isthmuses, and five 3-blocks.



**Fig.1.** 3-isthmuses and 3-blocks of a graph.

**Lemma 1** *Let  $S \subset V(G)$ . If  $|S| = k + 1$ , then there exists at most one  $k$ -block  $B$  of  $G$  such that  $S \subset B$ . Moreover, if  $|S| = k + 1$  and  $G[S]$  is connected, then there exists a unique  $k$ -block  $B$  of  $G$  such that  $S \subset B$ .*

**Proof.** Suppose toward a contradiction that  $B_1$  and  $B_2$  are two distinct  $k$ -blocks of  $G$  such that  $S \subset B_i$  for  $i = 1, 2$ . Let  $B = B_1 \cup B_2$ . Since  $G[B]$

is connected, by the maximality of  $B_2$ , there exists a  $k$ -isthmus  $I$  of  $G[B]$ . Hence, we have a partition  $B = X \cup Y \cup V(I)$  such that every path from  $X$  to  $Y$  is passing through  $I$ .

Since  $|I| < |S|$ , there exists a vertex  $u \in S \setminus I$ . We may assume  $u \in X$ . Take a vertex  $v \in Y$ . We may assume  $v \in B_1$  without loss of generality. Then both  $u$  and  $v$  are contained in  $B_1$ . Hence, by the connectivity of  $G[B_1]$ , we have  $I \subset B_1$ . Then  $I$  is a  $k$ -isthmus of  $G[B_1]$ , which contradicts that  $B_1$  is a  $k$ -block. Hence, there exists at most one  $k$ -block  $B$  such that  $S \subset B$ .

Suppose that  $G[S]$  is connected. Since  $G[S]$  has no  $k$ -isthmus, there exists a subset  $B \supset S$  such that (1)  $G[B]$  is connected, (2)  $G[B]$  has no  $k$ -isthmus of  $G[B]$  and (3)  $B$  is maximal with respect to (1) and (2). Then  $B$  is a  $k$ -block satisfying  $S \subset B$ .  $\square$

**Lemma 2** *Let  $S \subset V(G)$  with  $|S| \geq k+1$ . If  $G[S]$  is connected and  $G[S]$  has no  $k$ -isthmus of  $G[S]$ , then there exists a unique  $k$ -block  $B$  such that  $B \supset S$ .*

**Proof.** Take a family of  $(k+1)$ -subsets  $\{S_j\}_{j \in J}$  of  $V(G)$  such that  $S = \cup_{j \in J} S_j$  and  $G[S_j]$  is connected for  $j \in J$ . Then, by Lemma 1, there exists a family of  $k$ -blocks  $\{B_j\}_{j \in J}$  such that  $B_j \supset S_j$  for  $j \in J$ . Since  $S$  has no  $k$ -isthmus of  $G[S]$ , we have  $B_j \supset S$  for  $j \in J$ . Hence,  $B_j$  coincides with each other for  $j \in J$ . Let  $B = B_j$  for  $j \in J$ . Then  $B$  is a  $k$ -block containing  $S$ , as required.  $\square$

For a vertex  $v \in V(G)$ , let  $N(v)$  denote the set of neighbours of  $v$  in  $G$ . For a vertex set  $S \subset V(G)$ , let  $N(S) = \cup_{v \in S} N(v) \setminus S$ . For a pair of subsets  $T_1$  and  $T_2$  of  $V(G)$ , a path  $P$  of  $G$  is called a  $(T_1, T_2)$ -path if  $P$  begins from  $T_1$  and ends at  $T_2$  and all the inner vertices of  $P$  are not contained in  $T_1 \cup T_2$ .

**Lemma 3** *Let  $B_1$  and  $B_2$  be distinct  $k$ -blocks of a graph  $G$ . If there exists a  $k$ -subset  $S \subset B_1 \cap B_2$  such that  $G[S]$  is connected, then  $G[S]$  is a  $k$ -isthmus of  $G[B_1 \cup B_2]$  and  $G[S]$  is a  $k$ -isthmus of  $G$ .*

**Proof.** By the maximality of  $B_1$ , there exists a  $k$ -isthmus  $I = v_1 v_2 \dots v_k$  of  $G[B_1 \cup B_2]$ . We claim that  $V(I) = S$ . Suppose to a contradiction that  $V(I) \neq S$ . Then there exists a vertex  $u \in S \setminus V(I)$ , because  $|S| = |V(I)|$ . Since  $I$  is a  $k$ -isthmus of  $G[B_1 \cup B_2]$ , there exists a vertex  $v \in (B_1 \cup B_2) \setminus I$

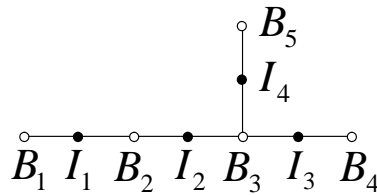


such that  $u$  and  $v$  are separated by  $I$ . By symmetry, we may assume  $v \in B_1$ . Since  $\{u, v\} \subset B_1$ , by the connectivity of  $G[B_1]$ , we have  $I \subset B_1$ . This implies that  $G[B_1]$  has a  $k$ -isthmus of itself, a contradiction. Therefore, we have  $V(I) = S$ .

Next, we show that the path  $I$  is a  $k$ -isthmus of  $G$ . Since  $I$  is a  $k$ -isthmus of  $G[B_1 \cup B_2]$ , there is a partition  $B_1 \cup B_2 = X \cup Y \cup V(I)$  such that  $I$  separates  $X$  and  $Y$  in  $G[B_1 \cup B_2]$ . Let  $Z = V(G) \setminus (B_1 \cup B_2)$ . Let  $u, v$  be two endvertices of  $I$  such that  $N(u) \cap X \neq \emptyset$  and  $N(v) \cap Y \neq \emptyset$ . Let  $X_1 = X \cup \{u\}$ ,  $Y_1 = Y \cup \{v\}$  and  $S_1 = V(I) \setminus \{u, v\}$ .

If there exists a vertex  $z \in Z$  such that  $z$  is adjacent to  $S_1$ , then  $B_1 \cup \{z\}$  has no isthmus of itself. This contradicts to the maximality of  $B_1$ . Hence, all  $(Z, B_1 \cup B_2)$ -paths end at  $X_1 \cup Y_1$ . If there exists a vertex  $z \in Z$  such that there exists a  $(z, X_1)$ -path and there exists a  $(z, Y_1)$ -path, then we have a  $(X_1, Y_1)$ -path  $P$  such that  $V(P) \cap S_1 = \emptyset$ . Then,  $G[B_1 \cup B_2 \cup V(P)]$  has no  $k$ -isthmus of itself. This contradicts to the maximality of  $B_1$ . Hence, we have a partition  $Z = X_2 \cup Y_2$  such that any  $(X_2, B_1 \cup B_2)$ -path ends at  $X_1$  and any  $(Y_2, B_1 \cup B_2)$ -path ends at  $Y_1$ . Let  $X' = X \cup X_2$  and  $Y' = Y \cup Y_2$ . Then we have a partition  $V(G) = X' \cup Y' \cup V(I)$  such that  $I$  separates  $X'$  and  $Y'$ . Therefore,  $I$  is a  $k$ -isthmus of  $G$ .  $\square$

For a connected graph  $G$ , let  $\mathcal{I}_k$  and  $\mathcal{B}_k$  denote the set of all  $k$ -isthmuses of  $G$  and the set of all  $k$ -blocks of  $G$ . The  $k$ -isthmus graph  $T_k$  of  $G$  is defined such that  $V(T_k)$  is  $\mathcal{B}_k \cup \mathcal{I}_k$  and  $E(T_k)$  is  $\{(B, I) \in \mathcal{B}_k \times \mathcal{I}_k : V(I) \subset B\}$ .



**Fig.2.** The isthmus graph of a graph  $G$  in Fig.1.

By definition, the isthmus graph is a bipartite graph. As a matter of fact, it is a tree.

**Proposition 4** *Let  $G$  be a connected graph. Then the  $k$ -isthmus graph  $T_k$  of  $G$  is a tree.*

**Proof.** Firstly, we show that  $T_k$  is connected. For a  $k$ -isthmus  $I$  of  $G$ , we can take a vertex  $u \in V(G) \setminus I$  such that  $G[I \cup \{u\}]$  is connected. By Lemma 1, we have a  $k$ -block  $B$  containing  $I$ . Hence,  $I$  is not isolated in  $T_k$ .

Let  $B_1$  and  $B_2$  be distinct  $k$ -blocks of  $G$ . We show that there exists a path from  $B_1$  to  $B_2$  in  $T_k$ . Take a  $(k+1)$ -subsets  $A_i$  of  $B_i$  such that  $G[A_i]$  is connected for  $i = 1, 2$ . Since  $G$  is connected, we have a finite sequence of  $(k+1)$ -sets  $A_1 = A'_1, A'_2, \dots, A'_s = A_2$  of  $V(G)$  such that  $G[A'_i]$  is connected for  $1 \leq i \leq s$  and  $|A'_i \cap A'_{i+1}| = k$ ,  $G[A'_i \cap A'_{i+1}]$  is connected for  $1 \leq i \leq s-1$ .

By Lemma 1, we have a sequence of  $k$ -block  $B'_i$  such that  $A'_i \subset B'_i$  for  $1 \leq i \leq s$ . Note that  $B_1 = B'_1$  and  $B_2 = B'_s$ . Furthermore, since  $|A'_i \cap A'_{i+1}| = k$  and  $G[A'_i \cap A'_{i+1}]$  is connected, by Lemma 3, we have  $B'_i = B'_{i+1}$  or  $B'_i$  and  $B'_{i+1}$  contains a common  $k$ -isthmus for  $1 \leq i \leq s-1$ . Hence, there is a path from  $B_1$  to  $B_2$  in  $T_k$ . Therefore,  $T_k$  is connected.

Secondly, we will show that  $T_k$  has no cycle. Suppose to a contradiction that  $T_k$  has a cycle  $C = I_1 B_1 I_2 \dots I_s B_s I_1$ , where  $I_i$  is a  $k$ -isthmus and  $B_i$  is a  $k$ -block for  $1 \leq i \leq s$  such that  $B_i \cap B_{i+1} = V(I_{i+1})$  for  $1 \leq i \leq s-1$  and  $B_s \cap B_1 = V(I_1)$ . Let us take a partition  $V(G) = X \cup Y \cup V(I_1)$  such that  $I_1$  separates  $X$  and  $Y$ . Since  $I_1$  is a  $k$ -isthmus contained in  $B_s \cap B_1$ , by Lemma 3, we may assume  $B_1 \subset X \cup V(I_1)$  and  $B_s \subset Y \cup V(I_1)$ . We claim that every  $k$ -block  $B$  of  $G$  satisfies  $B \subset X \cup V(I_1)$  or  $B \subset Y \cup V(I_1)$ . Indeed, otherwise we have  $B \cap X \neq \emptyset$  and  $B \cap Y \neq \emptyset$ . Since  $G[B]$  is connected, we have  $B \supset V(I_1)$ . It follows that  $I_1$  is a  $k$ -isthmus of  $B$ , a contradiction.

Then there exists an index  $\alpha$  with  $2 \leq \alpha \leq s$  such that  $B_{\alpha-1} \subset X \cup V(I_1)$  and  $B_\alpha \subset Y \cup V(I_1)$ . Since  $I_\alpha$  is a  $k$ -isthmus contained in  $B_{\alpha-1} \cap B_\alpha \subset V(I_1)$ , it coincides with  $I_1$ , a contradiction.  $\square$

We call  $T_k$  the *k-isthmus tree* of a graph  $G$ .

## 4 Contact Condition of Pebbles

Let  $G$  be a connected graph with  $n$  vertices. Let  $P$  be the set of pebbles of order  $m$ . For the configurations of  $P$  on  $G$ , let us define the *vacancy size*, denoted by  $k(P, G) := n - m$ , as the number of unoccupied vertices of  $G$ . In this section, let us use  $k$  in stead of  $k(P, G)$ , as an abbreviation. Because we treat only configurations which admit a move of a pebble, let us assume

$k \geq 1$  throughout this section.

For a configuration  $f \in \mathcal{F}$  and a pebble  $p \in P$ , let us define  $R(p, f) = \{v \in V(G) : g(p) = v \text{ for some configuration } g \text{ such that } g \sim f\}$ . We call  $R(p, f)$  the *reachable range* of  $p$  starting from  $f$ .

For a configuration  $f \in \mathcal{F}$  and a pebble  $p \in P$ , let  $v = f(p)$ . Since  $G$  is connected, we can gather  $k$  unoccupied vertices around  $v$  without moving  $p$ . More precisely, we have a configuration  $f_1$  equivalent to  $f$  such that (1)  $f_1(p) = v$ , (2)  $G[A]$  is connected where  $A = V(G) \setminus f_1(P \setminus \{p\})$ . Note that  $A$  is not uniquely determined by a given pair  $p \in P$  and  $f \in \mathcal{F}$ . Since  $|A| = k + 1$ , by Lemma 1 in the previous section, we have a  $k$ -block  $B$  of  $G$  containing  $A$ .

**Theorem 5** *Let  $p \in P$  and  $f \in \mathcal{F}$ . Let  $B$  be a  $k$ -block as defined as above. Then  $R(p, f) = B$ .*

**Proof.** Firstly, we will show that  $B \subset R(p, f)$ . Since  $G[B]$  is connected and  $G[B]$  has no  $k$ -isthmus of itself, by Theorem A and Theorem B,  $p$  can be moved to all the vertices of  $B$ . Hence, we have  $B \subset R(p, f)$ .

Secondly, we will show that  $R(p, f) \subset B$ . Suppose to a contradiction that  $R(p, f) \setminus B \neq \emptyset$ . Since  $G[R(p, f)]$  is connected, we may assume there exists a vertex  $u \in R(p, f) \setminus B$  such that  $G[B \cup \{u\}]$  is connected. Then by the maximality of  $B$ , there exists a  $k$ -isthmus  $I = v_1 v_2 \dots v_k$  of  $G[B \cup \{u\}]$ . Then we have  $V(I) \subset B$ . Let  $A = V(I) \cup \{u\}$ . Since  $G[A]$  is connected and  $|A| = k + 1$ , by Lemma 1, we have a  $k$ -block  $B'$  such that  $V(I) \cup \{u\} \subset B'$ . Then, by Lemma 3,  $I$  is a  $k$ -isthmus of  $G$ . We may assume  $u$  is a neighbour of  $v_k$ . Let  $C$  be the set of vertices of  $G$  separated by  $I$  from  $u$ . Hence, for any configuration  $g$  equivalent to  $f$  such that  $g(p) = v_i$  for some  $1 \leq i \leq k$ , the number of unoccupied vertices of  $C \cup \{v_1, v_2, \dots, v_{i-1}\}$  is at least  $i$ . In particular, if  $g \sim f$  and  $g(p) = v_k$ , then all the unoccupied vertices are contained in  $C \cup I$ . Therefore,  $p$  cannot be moved to  $u$ , a contradiction.  $\square$

Next, we consider contact condition. Let  $p, q$  be a pair of distinct pebbles. We say that  $p$  *contacts*  $q$  beginning from an initial configuration  $f$ , if there exists a configuration  $g$  equivalent to  $f$  such that  $g(p)$  and  $g(q)$  are adjacent.

If  $G$  is a cycle, it is easy to see that  $p$  contacts  $q$  if and only if  $q$  is next

to  $p$  along the cycle in the initial configuration. If  $k \geq 2$  and a  $k$ -block  $B$  is a proper subset of  $V(G)$ , then  $G[B]$  is not a cycle.

**Theorem 6** *Let  $p, q \in P$ , and  $f \in \mathcal{F}$ . Then  $p$  can contact  $q$ , if and only if one of the following conditions hold; (1)  $R(p, f) = R(q, f)$  and  $G[R(p, f)]$  is not a cycle, or (2)  $R(p, f) = R(q, f)$  and  $G[R(p, f)]$  is a cycle and  $q$  is next to  $p$  along the cycle, or (3)  $R(p, f) \neq R(q, f)$  and  $G[R(p, f) \cap R(q, f)]$  is a  $k$ -isthmus of  $G$ .*

**Proof.** Firstly, we show that any of the conditions (1), (2) or (3) is sufficient for the contact. Suppose that  $R(p, f) = R(q, f)$ . Let  $B = R(p, f)$ . By Theorem 5,  $B$  is a  $k$ -block of  $G$ . Then we have a configuration  $f_1$  equivalent to  $f$  such that  $V(G) \setminus f_1(P \setminus \{p\}) \subset B$ . Since  $R(p, f) = R(q, f)$ , we have  $V(G) \setminus f_1(P \setminus \{p, q\}) \subset B$ . Let us consider the puzzle on the restricted graph  $G[B]$ . Let  $m'$  be the number of pebbles on  $B$  in  $f_1$ . It is easy to see that if  $G[B]$  is a cycle and  $q$  is next to  $p$  along the cycle,  $p$  can contact  $q$ . Suppose that  $G[B]$  is not a cycle. By Theorem A and Theorem B, the puzzle  $\text{puz}(G[B], k)$  is 3-transitive for  $m' \geq 3$ . For  $m' = 2$ , since  $G[B]$  is not a cycle,  $\text{puz}(G[B], k)$  is feasible. Hence, in all the cases,  $p$  can contact  $q$ .

Next, we assume that  $R(p, f) \neq R(q, f)$  and  $I = G[R(p, f) \cap R(q, f)]$  is a  $k$ -isthmus  $I$  of  $G$ . Let us take two vertices  $u \in (R(p, f) \setminus I) \cap N(I)$  and  $v \in (R(q, f) \setminus I) \cap N(I)$ . Since  $\text{puz}(G[R(p, f)], k)$  is feasible, there exists a configuration  $f_1$  equivalent to  $f$  such that  $f_1(p) = u$ , and  $V(G) \setminus f_1(P) = I$ . Then since the puzzle  $\text{puz}(G[R(q, f_1)], k)$  is feasible, there exists a configuration  $f_2$  equivalent to  $f_1$  such that  $f_2(p) = u$ ,  $f_2(q) = v$ , and  $V(G) \setminus f_2(P) = I$ . Then by using a path  $I$ ,  $p$  can contact  $q$ .

Secondly, we show that one of the conditions (1), (2) or (3) is necessary for the contact. Let  $f_1$  be a configuration equivalent to  $f$  such that  $u = f_1(p)$  and  $v = f_1(q)$ , where  $u$  and  $v$  are adjacent. Let us gather  $k$  unoccupied vertices around  $u, v$  without moving  $p$  and  $q$ . More precisely, we have a configuration  $f_2$  equivalent to  $f_1$  such that (1)  $f_2(p) = u$ , (2)  $f_2(q) = v$ , (3)  $G[A]$  is connected where  $A = V(G) \setminus f_2(P \setminus \{p, q\})$ . If  $G[A]$  has no  $k$ -isthmus of  $G$ , by Lemma 2, we have a unique  $k$ -block  $B$  containing  $A$ . In this case, by Theorem 5, we have  $B = R(p, f) = R(q, f)$ . If  $G[R(p, f)]$  is a cycle and  $q$  is not next to  $p$  along the cycle,  $p$  cannot contact  $q$ .

Suppose that  $G[A]$  has a  $k$ -isthmus  $I$  of  $G$ . In this case,  $G[A]$  is a path of  $G$ . Let  $x_1, x_2$  be the two endvertices of  $G[A]$ . Note that  $A \setminus \{x_1, x_2\} = V(I)$ .

Then there exists a configuration  $f_3$  equivalent to  $f_2$  such that  $f_3(\{p, q\}) = \{x_1, x_2\}$  and  $f_3^{-1}(I) = \emptyset$ . Then we have  $G[R(p, f) \cap R(q, f)] = I$ , as required.  $\square$

Lastly, we consider a necessary and sufficient condition such that the two configurations  $f$  and  $g$  are equivalent.

First, we deal with the case  $k \geq 2$ .

**Theorem 7** *Let  $k \geq 2$ . Let  $f$  and  $g$  be two configurations. Then  $f$  and  $g$  are equivalent if and only if all the following conditions hold;*

- (1)  $R(p, f) = R(p, g)$  for any pebble  $p \in P$ .
- (2) If  $G$  is a cycle graph, then the cyclic order of  $P$  on  $G$  is the same as in  $f$  and  $g$ .

**Proof.** Suppose that  $f$  and  $g$  are equivalent. Then any pebble  $p$  on  $f(p)$  with a configuration  $f$  can be moved at  $g(p)$  with  $g$ , and vice versa. Hence we have  $R(p, f) = R(p, g)$ . It is not difficult to check the condition (2).

Conversely, suppose that the conditions (1) and (2) hold. We proceed by induction on the number  $s$  of  $k$ -blocks of  $G$ . Let  $s = 1$ . Since  $G$  has no  $k$ -isthmus, by Theorem B, if  $G$  is not a cycle,  $(G, k)$  is feasible. Hence,  $f$  and  $g$  are equivalent. If  $G$  is a cycle, by the condition (2),  $f$  and  $g$  are equivalent.

Let  $s \geq 2$ . In this case, note that any  $k$ -block of  $G$  is not a cycle, because it contains a set of vertices of some  $k$ -isthmus. Let us take a  $k$ -block  $B$  such that  $B$  corresponds to a leaf of the  $k$ -isthmus tree of  $G$ . Let  $I$  be a unique  $k$ -isthmus such that  $B \supset V(I)$ . Then we have two configurations  $f_1$  and  $g_1$  such that  $f_1 \sim f$ ,  $g_1 \sim g$  and  $f_1^{-1}(V(I)) = g_1^{-1}(V(I)) = \emptyset$ . Put  $P_1 = f_1^{-1}(B)$ . Then, for  $p \in P$ , we have  $p \in P_1$  if and only if  $R(p, f) = B$ . By the condition (1), we have  $P_1 = g_1^{-1}(B)$ . Since  $(G[B].k)$  is feasible,  $f_1|_{P_1}$  and  $g_1|_{P_1}$  are equivalent. Now, we have two configurations  $f_2$  and  $g_2$  such that  $f_2 \sim f$ ,  $g_2 \sim g$ , and  $f_2(p) = g_2(p) \in B \setminus V(I)$  for all  $p \in P_1$ . Let us define  $G' = G \setminus (B \setminus V(I))$  and  $P' = P \setminus P_1$ . Then  $G'$  has  $s - 1$   $k$ -isthmuses. By inductive hypothesis,  $f_2|_{P'}$  and  $g_2|_{P'}$  are equivalent on  $G'$ . Therefore,  $f$  and  $g$  are equivalent on  $G$ .  $\square$

Next, we deal with the case  $k = 1$ . In this case, according to Theorem A, the conditions for equivalence becomes a little complicated.

**Theorem 8** *Let  $k = 1$ . Let  $f$  and  $g$  be two configurations. Then  $f$  and  $g$  are equivalent if and only if all the following conditions hold;*

- (1)  $R(p, f) = R(p, g)$  for any pebble  $p \in P$ .
- (2) For  $x \in V(G)$ , let  $f_x$  and  $g_x$  be arbitrary configurations such that  $f_x \sim f$ ,  $g_x \sim g$  and  $f_x^{-1}(x) = g_x^{-1}(x) = \emptyset$ .
  - (2-i) Suppose that  $G[R(p, f)]$  is a cycle graph for a pebble  $p$ . Let  $x \in R(p, f)$ . Then  $f_x^{-1}(y) = g_x^{-1}(y)$  for all  $y \in R(p, f) \setminus \{x\}$ .
  - (2-ii) Suppose that  $G[R(p, f)]$  is a bipartite graph for a pebble  $p$ . Let  $x \in R(p, f)$ . Then,  $g_x \circ f_x^{-1}$  restricted on  $R(p, f) \setminus \{x\}$  is an even permutation.
  - (2-iii) Suppose that  $G[R(p, f)]$  is the  $\theta(1, 2, 2)$  graph for a pebble  $p$ . Let  $x \in R(p, f)$ . Then,  $g_x \circ f_x^{-1}$  restricted on  $R(p, f) \setminus \{x\}$  is contained in  $PGL_2(5)$ , which is the projective general linear group on 2-dimensional vector space over a finite field of order 5.

Theorem 8 is proved in a similar manner as in the proof of Theorem 7, and we omit the proof.

## 5 Equivalence of Arrangements

In this section, we focus on the case of AAP. The proof for the case of SGA is almost identical to the case of AAP and will be omitted.

**Lemma 9** *Let  $(G_A, G_M)$  be an ordered pair of graphs, let  $\{ps, t\}$  be an edge of  $G_M$ , and let  $f$  be an arrangement of  $(G_A, G_M)$  such that  $|f^{-1}(\{s, t\})| \geq 2$  and the graph  $G_A[f^{-1}(\{s, t\})]$  is connected. Then the arrangement  $f$  is  $t$ -equivalent (in the sense of AAP) to the following arrangement  $g$ :*

$$g(x) = \begin{cases} t, & \text{if } f(x) \in \{s, t\}; \\ f(x), & \text{if } f(x) \notin \{s, t\}. \end{cases}$$

Note that, as for the case of SGA, by the definition, the conclusion of this lemma holds even if we omit the assumption “ $|f^{-1}(\{s, t\})| \geq 2$ ” for the arrangement  $f$  from its statement.

**Proof.** We divide our proof into two cases.

**Case 1**  $f^{-1}(t) \neq \emptyset$ .

Let  $U := f^{-1}(s)$ . Because  $f^{-1}(\{s, t\}) \setminus U = f^{-1}(t) \neq \emptyset$  and  $G_A[U]$  is connected, by the definition of transfer of AAP, the arrangement  $f$  is t-equivalent to the arrangement  $g$ .

**Case 2**  $f^{-1}(t) = \emptyset$ .

In this case, because  $f^{-1}(s) \geq 2$  and  $G_A[f^{-1}(s)]$  is connected, there exists at least one vertex  $u$  in  $f^{-1}(s)$  such that  $G_A[f^{-1}(s)] - u$  remains connected. Thus, by the definition of transfer of AAP, the arrangement  $f$  is t-equivalent to the following arrangement  $f'$ :

$$f'(x) = \begin{cases} t, & \text{if } x = u; \\ f(x), & \text{if } x \neq u. \end{cases}$$

This  $f'$  satisfies the condition of **Case 1**, and hence  $g \cong f' \cong f$  holds.

□

Let  $f$  be an arrangement of an ordered pair  $(G_A, G_M)$  of graphs. Lemma 9 indicates that, if a country  $s$  of the route map  $G_M$  contains at least two agents under the arrangement  $f$ , we can treat the agent sub-network  $G_A[f^{-1}(s)]$  as if a single ‘pebble’ on the vertex  $s$  of the ‘board graph’  $G_M$ . This observation leads us to define the following set of new concepts for the AAP model. For every arrangement  $f$  of the ordered pair  $(G_A, G_M)$ , let us define the *isolated-agent set*  $IA_f$  as the set of agents  $\{a \in V(G_A) : f^{-1}(f(a)) = \{a\}\}$ . In other words, the set  $IA_f$  is the set of all agents who are staying alone in their countries under the arrangement  $f$ . In the same way, let us define the *isolated-country set*  $IC_f$  as the set of countries  $\{c \in V(G_M) : |f^{-1}(c)| = 1\}$ . Note that  $f(IA_f) = IC_f$  holds. Now, let us consider the following configuration  $\phi_f$  of the pebble motion problem corresponding to the arrangement  $f$ : The pebble set  $P_{\phi_f}$  for  $\phi_f$  is the set  $\{f^{-1}(c) : c \in V(G_M) \setminus IC_f, f^{-1}(c) \neq \emptyset\}$ . The board graph  $G_{\phi_f}$  for  $\phi_f$  is the subgraph of  $G_M$  induced by the vertex set  $V(G_M) \setminus IC_f$ . Then our configuration is defined by  $\phi_f : P_{\phi_f} \ni f^{-1}(c) \mapsto c \in V(G_{\phi_f})$ . Let us call this  $\phi_f$  the *configuration associated with  $f$* . For two arrangements  $f$  and  $g$  of  $(G_A, G_M)$ , we say that the configuration  $\phi_f$  associated with  $f$

is *equivalent* to the configuration  $\phi_g$  associated with  $g$  and use the notation  $\phi_f \sim \phi_g$ , if and only if both their pebble sets and board graphs are coincident with each other ( $P_{\phi_f} = P_{\phi_g}$  and  $G_{\phi_f} = G_{\phi_g}$ ) and  $\phi_f$  is equivalent to  $\phi_g$  in the sense of the pebble motion problem. It is clear that  $\phi_f \sim \phi_g$  implies  $f \cong g$ . Although the converse is not true in general, the following extremal condition warrants its affirmation:

Let  $f$  be an arrangement of an ordered pair  $(G_A, G_M)$  of graphs. Let  $\phi_f$  denote the configuration associated with  $f$ ,  $P_{\phi_f}$  its pebble set,  $G_{\phi_f}$  its board graph. Now let us define the set of *f-irreducible arrangements*  $\text{Irr}(f) := \{h : h \cong f, |P_{\phi_h}| = \min\{|P_{\phi_g}| : g \cong f\}\}$  and the set of *f-irreducible configurations*  $\Phi(\text{Irr}(f)) := \{\phi_h : h \in \text{Irr}(f)\}$ .

**Theorem 10** *For an arbitrary arrangement  $f$  of an arbitrary ordered pair  $(G_A, G_M)$  of graphs, all configurations in the set  $\Phi(\text{Irr}(f))$  are equivalent.*

In order to prove the above theorem, we shall prepare some notations and prove a technical theorem.

Let  $(G_A, G_M)$  be an ordered pair of graphs,  $\{s, t\}$  be an edge of  $G_M$ , and let  $f$  be an arrangement of  $(G_A, G_M)$  such that  $f^{-1}(s) \neq \emptyset$ ,  $|f^{-1}(\{s, t\})| \geq 2$  and the graph  $G_A[f^{-1}(\{s, t\})]$  is connected. Lemma 9 guarantees that the following arrangement  $g$  can be achieved from the arrangement  $f$  by at most two steps of transfers (in the sense of AAP):

$$g(x) = \begin{cases} t, & \text{if } f(x) \in \{s, t\}; \\ f(x), & \text{if } f(x) \notin \{s, t\}. \end{cases}$$

Let us call the sequence of (at most two) transfers from the arrangement  $f$  to the arrangement  $g$  a *SUBNET MERGER*. Furthermore, let us call a SUBNET MERGER from the arrangement  $f$  to the arrangement  $g$  a *SUBNET MOVE* if the set  $f^{-1}(t)$  is empty. A SUBNET MERGER is called *proper* if it is not a SUBNET MOVE.

**Theorem 11** *Let  $f$  be an arbitrary arrangement of an arbitrary ordered pair  $(G_A, G_M)$  of graphs. Then, for every  $f$ -irreducible arrangement  $g$ , there exists a sequence  $f =: f_0 \cong f_1 \cong \dots \cong f_k := g$  of  $t$ -equivalent arrangements on  $(G_A, G_M)$  such that, for all  $i \in \{0, \dots, k-1\}$ , each sequence of the transfers from the arrangement  $f_i$  to the arrangement  $f_{i+1}$  is a SUBNET MERGER.*



**Proof of Theorem 11.** Because  $g$  is an  $f$ -irreducible arrangement, there exists a sequence  $f =: f_0 \cong f_1 \cong \dots \cong f_k := g$  of  $t$ -equivalent arrangements such that, for all  $i \in \{0, \dots, k-1\}$ , the sequence of the transfers from  $f_i$  to  $f_{i+1}$  is either a SUBNET MERGER or consisting of a single transfer which is not a SUBNET MERGER. For the above sequence  $f_0 \cong f_1 \cong \dots \cong f_k$ , let  $m$  be the minimum number in  $\{0, \dots, k\}$  on condition that the sequence of the transfers from  $f_m$  to  $f_k (= g)$  can be written as a sequence of only SUBNET MERGERS. We will show that  $m = 0$  by reductio ad absurdum.

Because of the minimality of  $m$ , we can assume that each arrangement  $f_{i+1}$  ( $i = m, \dots, k-1$ ) can be achieved from the arrangement  $f_i$  by a SUBNET MERGER. And hence we have that;

$$\forall i \in \{m, \dots, k-1\}, \exists \{s_i, t_i\} \in E(G_M), f_{i+1}(x) = \begin{cases} t_i, & \text{if } f_i(x) \in \{s_i, t_i\}; \\ f_i(x), & \text{if } f_i(x) \notin \{s_i, t_i\}. \end{cases}$$

Now suppose that  $m \geq 1$ . Since  $m$  is minimum, we have that the sequence of the transfers from the arrangement  $f_{m-1}$  to the arrangement  $f_m$  is consisting of a single transfer which is not a SUBNET MERGER. That is, there exist an edge  $\{s_{m-1}, t_{m-1}\}$  of  $G_M$  and a proper non-empty subset  $U_{m-1}$  of the set  $f_{m-1}^{-1}(s_{m-1})$  (i.e.  $\emptyset \neq U_{m-1} \subsetneq f_{m-1}^{-1}(s_{m-1})$ ) such that:

$$f_m(x) = \begin{cases} t_{m-1}, & \text{if } x \in U_{m-1}; \\ f_{m-1}(x), & \text{otherwise.} \end{cases}$$

Now let  $h_1$  be the following arrangement:

$$h_1(x) := \begin{cases} t_{m-1}, & \text{if } f_{m-1}(x) \in \{s_{m-1}, t_{m-1}\}; \\ f_{m-1}(x), & \text{if } f_{m-1}(x) \notin \{s_{m-1}, t_{m-1}\}. \end{cases}$$

Clearly, this  $h_1$  can be achieved from the arrangement  $f_{m-1}$  by a SUBNET MERGER. Let  $\mathcal{H}$  be the set of all arrangements achieved from  $h_1$  by a sequence of only SUBNET MERGERS. And let  $g'$  be an arrangement in  $\mathcal{H}$  such that  $|g'(V(G_A))| = \min\{|h(V(G_A))| : h \in \mathcal{H}\}$  holds. Let  $f_{m-1} =: h_0 \cong h_1 \cong \dots \cong h_l := g'$  be a sequence of arrangements such that, for all  $i \in \{1, \dots, l\}$ ,  $h_i$  can be achieved from  $h_{i-1}$  by a single SUBNET MERGER. Then we have that;

$$\forall i \in \{0, \dots, l-1\}, \exists \{u_i, v_i\} \in E(G_M), h_{i+1}(x) = \begin{cases} v_i, & \text{if } h_i(x) \in \{u_i, v_i\}; \\ h_i(x), & \text{if } h_i(x) \notin \{u_i, v_i\}. \end{cases}$$

Because of the minimality of  $m$ , we have that  $g \neq g'$ . Furthermore, since  $g$  is an  $f$ -irreducible arrangement,  $|g'(V(G_A))| \geq |g(V(G_A))|$  holds. And hence

there exist two distinct agents  $a, b \in V(G_A)$  such that both  $g(a) = g(b)$  and  $g'(a) \neq g'(b)$  hold. Now let us number all the vertices of  $G_A$  so that  $V(G_A) := \{a_1 := a, a_2 := b, \dots, a_n\}$ . Let  $\text{id}(S) := \min\{i : a_i \in S\}$  be the function from the power set of  $V(G_A)$  to the set  $\{1, \dots, n\}$  which returns the minimum index number of vertices in a given subset  $S$  of  $V(G_A)$ . Corresponding to the above sequence of arrangements  $f_{m-1} = h_0 \cong h_1 \cong \dots \cong h_l = g'$ , we will define another sequence of arrangements  $g' =: h'_l \cong h'_{l-1} \cong \dots \cong h'_0$  such that, for all  $i \in \{0, \dots, l-1\}$ ,

$$h'_i(x) := \begin{cases} u_i, & \text{if } h'_{i+1}(x) = v_i \text{ and } h_i(a_{\text{id}(h'_{i+1}^{-1}(v_i))}) = u_i; \\ h'_{i+1}(x), & \text{otherwise.} \end{cases}$$

Because each arrangement  $h_{i+1}$  ( $i = 0, \dots, l-1$ ) is achieved from  $h_i$  by a SUBNET MERGER, the corresponding sequence of the transfers from  $h'_{i+1}$  to  $h'_i$  defined above is a SUBNET MOVE. And hence, for all  $i \in \{0, \dots, l\}$ ,  $h'_i \in \mathcal{H}$  and  $|h'_i(V(G_A))| = |g'(V(G_A))|$ . Here we note that, there exists another representation of the arrangements  $h'_i$  ( $i = 0, \dots, l$ ), as follows:

$$\forall(i, x) \in \{0, \dots, l\} \times V(G_A), h'_i(x) = h_i(a_{\text{id}(g'^{-1}(g'(x)))}).$$

Let  $f'_m$  be the following arrangement:

$$f'_m(x) := \begin{cases} t_{m-1}, & \text{if } h'_0(x) = s_{m-1} \text{ and } f_m(a_{\text{id}(h'_0^{-1}(s_{m-1}))}) = t_{m-1}; \\ h'_0(x), & \text{otherwise.} \end{cases}$$

Because  $h'_0$  is achieved from  $f_{m-1}$  ( $= h_0$ ) by a sequence of only SUBNET MERGERS, for every country  $c \in V(G_M)$ , if  $f_{m-1}^{-1}(c) \neq \emptyset$  then  $f_{m-1}^{-1}(c) \subseteq h'_0^{-1}(c)$  holds. In particular, if  $f_{m-1}^{-1}(s_{m-1}) \neq \emptyset \neq f_{m-1}^{-1}(t_{m-1})$ , then  $f_{m-1}^{-1}(s_{m-1}) \subseteq h'_0^{-1}(s_{m-1})$  and  $f_{m-1}^{-1}(t_{m-1}) \subseteq h'_0^{-1}(t_{m-1})$  hold, and the graph  $G_A[h'_0^{-1}(\{s_{m-1}, t_{m-1}\})]$  is connected. Then  $f'_m$  is achieved from  $h'_0$  by a SUBNET MERGER and  $|f'_m(V(G_A))| = |h'_0(V(G_A))| - 1 = |g'(V(G_A))| - 1$  holds, which contradicts the minimality of the size  $|g'(V(G_A))|$ . Hence we have that at least one of  $f_{m-1}^{-1}(s_{m-1})$  or  $f_{m-1}^{-1}(t_{m-1})$  is empty, and that  $f'_m$  is achieved from  $h'_0$  by a SUBNET MOVE. Furthermore,  $f'_m$  turns to be achieved from  $g'$  by a sequence of only SUBNET MOVES. Now, let  $f'_{m-1} := h'_0$  and we will define one more sequence of arrangements  $h'_0 = f'_{m-1} \cong f'_m \cong \dots \cong f'_k$  such that, for all  $i \in \{m, \dots, k\}$ ,

$$f'_i(x) := \begin{cases} t_{i-1}, & \text{if } f'_{i-1}(x) = s_{i-1} \text{ and } f_i(a_{\text{id}(f'_{i-1}^{-1}(s_{i-1}))}) = t_{i-1}; \\ f'_{i-1}(x), & \text{otherwise.} \end{cases}$$

Again, we have another representation of the arrangements  $f'_i$  ( $i = m, \dots, k$ ), as follows:

$$\forall(i, x) \in \{m, \dots, k-1\} \times V(G_A), f'_i(x) = f_i(a_{\text{id}(g'^{-1}(g'(x)))}).$$

Because  $f'_m$  is achieved from  $f_{m-1}$  by a sequence of only SUBNET MERGERS, for every country  $c \in V(G_M)$ , if  $f_{m-1}^{-1}(c) \neq \emptyset$  then  $f_{m-1}^{-1}(c) \subseteq f'_m{}^{-1}(c)$  holds. Then because each arrangement  $f_{i+1}$  ( $i = m, \dots, k-1$ ) is achieved from  $f_i$  by a SUBNET MERGER, the corresponding sequence of the transfers from  $f'_i$  to  $f'_{i+1}$  defined above is also a SUBNET MERGER. Combining this fact with the minimality of the size  $|g'(V(G_A))|$ , we have that each sequence of the transfers from  $f'_i$  ( $i = m, \dots, k-1$ ) to  $f'_{i+1}$  defined above is not only a SUBNET MERGER but also a SUBNET MOVE. Then  $f'_k$  turns out to be achieved from  $g'$  by a sequence of only SUBNET MOVES, and hence  $f'_k(a) \neq f'_k(b)$ . However it contradicts the fact that

$$\begin{aligned} f'_k(a) &= f_k(a_{\text{id}(g'^{-1}(g'(a)))}) = f_k(a_{\text{id}(g'^{-1}(g'(a_1)))}) \\ &= f_k(a_1) = g(a_1) = g(a) = g(b) = g(a_2) = f_k(a_2) \\ &= f_k(a_{\text{id}(g'^{-1}(g'(a_2)))}) = f_k(a_{\text{id}(g'^{-1}(g'(b)))}) \\ &= f'_k(b). \end{aligned}$$

Hence we have that  $g = g'$  and  $m = 0$ , which completes the proof.  $\square$

**Proof of Theorem 10.** Let  $g, h$  be two arbitrary  $f$ -irreducible arrangements,  $\phi_g, \phi_h$  their corresponding  $f$ -irreducible configurations. We will show that  $\phi_g \sim \phi_h$ .

Because  $g$  and  $h$  are  $f$ -irreducible arrangements, from Theorem 11, we have that there exists two sequences of arrangements  $f =: g_0 \cong g_1 \cong \dots \cong g_k := g$  and  $f =: h_0 \cong h_1 \cong \dots \cong h_l := h$  such that each  $g_{i+1}$  ( $i = 0, \dots, k-1$ ) (resp.  $h_{j+1}$  ( $j = 0, \dots, l-1$ )) is achieved from  $g_i$  (resp.  $h_j$ ) by a single SUBNET MERGER. In the same way of the proof of Theorem 11, let us number all the vertices of  $G_A$  as  $V(G_A) := \{a_1, \dots, a_n\}$  and define the function  $\text{id}(S) := \min\{i : a_i \in S\}$ . Corresponding to the above sequence of arrangements  $f = g_0 \cong g_1 \cong \dots \cong g_k = g$ , we will define another new sequence of arrangements  $g =: g'_k \cong g'_{k-1} \cong \dots \cong g'_0$  as follows:

$$\forall(i, x) \in \{0, \dots, k\} \times V(G_A), g'_i(x) := g_i(a_{\text{id}(g'^{-1}(g'(x)))}).$$

Because each arrangement  $g_{i+1}$  ( $i = 0, \dots, k-1$ ) is achieved from  $g_i$  by a SUBNET MERGER, the corresponding sequence of the transfers from  $g'_{i+1}$

to  $g'_i$  defined above is a SUBNET MOVE. Then next, corresponding to the above sequence of arrangements  $f = h_0 \cong h_1 \cong \dots \cong h_l = h$ , we will define the following new sequence of arrangements  $g'_0 =: h'_0 \cong h'_1 \cong \dots \cong h'_l$ :

$$\forall(j, x) \in \{0, \dots, l\} \times V(G_A), \quad h'_j(x) := h_j(a_{\text{id}(g'^{-1}(g'(x)))}).$$

Again, because the arrangement  $g'_0 (= h'_0)$  is achieved from the arrangement  $g_k (= g)$  by a sequence of only SUBNET MOVEs, and because each arrangement  $h_{i+1}$  ( $i = 0, \dots, l-1$ ) is achieved from  $h_i$  by a SUBNET MERGER, the corresponding sequence of the transfers from  $h'_{i+1}$  to  $h'_i$  defined above is a SUBNET MERGER. Furthermore, because  $g$  is an  $f$ -irreducible arrangement, the sequence of the transfers from  $h'_{i+1}$  to  $h'_i$  is not only a SUBNET MERGER, but also a SUBNET MOVE. And hence the arrangement  $h'_l$  is achieved from the arrangement  $g$  by a sequence of only SUBNET MOVEs. This fact tells us that, for every country  $c \in V(G_M)$ , if  $g^{-1}(c) \neq \emptyset$  then  $g^{-1}(c) = h'^{-1}_l(h(a_{\text{id}(g^{-1}(c))})) \supseteq h^{-1}(h(a_{\text{id}(g^{-1}(c))}))$  holds. Because we can choose an arbitrary numbering for the vertices of  $G_A$ , for an arbitrary country  $d$  such that  $h^{-1}(d) \neq \emptyset$  holds, we can assume that  $a_1 \in h^{-1}(d)$ . Combining this observation with the previous fact, we have that, for all  $d \in V(G_M)$ , if  $h^{-1}(d) \neq \emptyset$  then there exists a country  $c \in V(G_M)$  such that  $h^{-1}(d) \subset g^{-1}(c)$ . By the symmetry of the roles of  $g$  and  $h$ , we have that  $\{g^{-1}(c) | c \in V(G_M), g^{-1}(c) \neq \emptyset\} = \{h^{-1}(d) | d \in V(G_M), h^{-1}(d) \neq \emptyset\}$ , and hence  $h'_l = h$ , which means that  $\phi_g \sim \phi_h$ .  $\square$

Theorem 11 has a corollary, which plays a key role with Theorem 10 in our next algorithm for deciding t-equivalence.

**Corollary 12** *Let  $(G_A, G_M)$  be an arbitrary ordered pair of graphs, let  $f$  be an arbitrary arrangement of  $(G_A, G_M)$ , let  $g$  be an arbitrary  $f$ -irreducible arrangement, and let  $h$  be an arbitrary arrangement which is t-equivalent to  $f$ . Then  $g$  can be achieved from  $h$  by a sequence of only SUBNET MERGERs.*

**Proof.** It is derived from Theorem 11 and the fact that every  $f$ -irreducible arrangement is also  $h$ -irreducible.  $\square$

Theorem 10 tells us that two arrangements  $f$  and  $g$  are t-equivalent if and only if at least one  $f$ -irreducible configuration is equivalent to at least one  $g$ -irreducible configuration. Thanks to Corollary 12, in order to find an  $f$ -irreducible arrangement, starting from the initial arrangement  $f$ , we can

take an arbitrary proper SUBNET MERGER for the current arrangement and ‘contract’ the pebbles of its corresponding configuration, iteratively. By using Theorem 6, we can find a proper SUBNET MERGER for a given arrangement in polynomial-time. If there exists no proper SUBNET MERGER, Corollary 12 tells us that the current arrangement is  $f$ -irreducible. If we obtain an  $f$ -irreducible arrangement  $f_{\text{Irr}}$  and an  $g$ -irreducible arrangement  $g_{\text{Irr}}$ , by using Theorems 7 and 8, we can check whether the corresponding two configurations  $\phi_{f_{\text{Irr}}}$  and  $\phi_{g_{\text{Irr}}}$  are equivalent or not in polynomial-time. Combining these observations, we have the following algorithm which decides in polynomial-time whether given two arrangements of an ordered pair of graphs are t-equivalent or not.

**Algorithm 5.1** (**t-equivalenceDecision** $((G_A, G_M), f, g)$ )

**INPUT** : Two arrangements  $f, g$  of an ordered graph pair  $(G_A, G_M)$ .

**OUTPUT** : Decide whether  $f$  and  $g$  are t-equivalent or not.

- ① Let  $\phi_{f_{\text{Irr}}} := \text{IrreducibleConfiguration}((G_A, G_M), f)$  and let  $\phi_{g_{\text{Irr}}} := \text{IrreducibleConfiguration}((G_A, G_M), g)$ .
- ② **If**  $\phi_{f_{\text{Irr}}} \sim \phi_{g_{\text{Irr}}}$  **then return** YES, otherwise **return** NO.

**Algorithm 5.2** (**IrreducibleConfiguration** $((G_A, G_M), f)$ )

**INPUT** : An arrangement  $f$  of an ordered graph pair  $(G_A, G_M)$ .

**OUTPUT** : An  $f$ -irreducible configuration  $\phi_{f_{\text{Irr}}}$ .

- ① Set  $f_0 := f$ . Make the isolated-agent set  $\text{IA}_0$  of  $f_0$ . Make the configuration  $\phi_0$  associated with  $f_0$ . Let  $G_0$  denote the board graph of  $\phi_0$ , let  $P_0$  denote the pebble set for  $\phi_0$ , and let  $l_0$  denote the number of connected components of  $G_0$ .
- ② **If**  $\text{ContractiblePair}(\phi_i) = \{p, q\}$  **then goto** ③; **else goto** ⑥.

③ Set a new arrangement  $f_{i+1}$  as

$$f_{i+1}(x) := \begin{cases} \phi_i(p), & \text{if } x \in p \cup q; \\ f_i(x), & \text{if } x \notin p \cup q. \end{cases}$$

④ Make the isolated-agent set  $\text{IA}_{i+1}$  of  $f_{i+1}$  by modifying the set  $\text{IA}_i$ . Make the configuration  $\phi_{i+1}$  associated with  $f_{i+1}$  by modifying the configuration  $\phi_i$ . Let  $G_{i+1}$  denote the board graph of  $\phi_{i+1}$ , let  $P_{i+1}$  denote the pebble set for  $\phi_{i+1}$ , and let  $l_{i+1}$  denote the number of connected components of  $G_{i+1}$ .

⑤ Set  $i := i + 1$  and **goto** ②.

⑥ **Return** the configuration  $\phi_i$  associated with the arrangement  $f_i$ .

### Algorithm 5.3 (ContractiblePair( $\phi_i$ ))

**INPUT** : The isolated-agent set  $\text{IA}_i$  and the configuration  $\phi_i$  defined in **Algorithm 5.2**.

**OUTPUT** : A subset  $\{p, q\}$  of  $P_i \cup \text{IA}_i$  such that  $p$  can contact  $q$  and the graph  $G_A[p \cup q]$  is connected, if any. Otherwise  $\emptyset$ .

① **If** there exists a pair of isolated-agents  $\{a, b\} (\subseteq \text{IA}_i)$  such that the pair  $\{a, b\}$  is an edge of  $G_A$  and that the pair  $\{f_i(a), f_i(b)\}$  is an edge of  $G_M$ , **then return**  $\{\{a\}, \{b\}\}$ .

② Let  $G_{i,j} (j = 1, \dots, l_i)$  denote each connected component of  $G_i$ . Let  $P_{i,j} := \phi_i^{-1}(V(G_{i,j}))$  and let  $\phi_{i,j}$  denote the configuration of the pebble set  $P_{i,j}$  on the board graph  $G_{i,j}$  as the restriction of  $\phi_i$ .

**For**  $j := 1$  to  $l_i$  **do**:

(②-a) Set  $k_j := |V(G_{i,j})| - |P_{i,j}|$ , the vacancy size of the configuration  $\phi_{i,j}$ . Make the  $k_j$ -isthmus tree  $T_{k_j}(G_{i,j})$  of the board graph  $G_{i,j}$  of the configuration  $\phi_{i,j}$ . Set  $T := T_{k_j}(G_{i,j})$ .

- (②-b) Choose a leaf  $u$  of  $T$ . Let  $B_u$  be a  $k_j$ -block of  $G_{i,j}$  corresponding to  $u$ . **If**  $T$  has a vertex  $v$  such that the length of the (unique) path of  $T$  from  $u$  to  $v$  is 2 **then** let  ${}_uI_v$  denote the middle vertex of the path, and let  $B_v$  denote a  $k_j$ -block of  $G_{i,j}$  corresponding to  $v$  **else**  $B_v := \emptyset$ . Let  $IC_i$  denote the isolated-country set of  $f_i$ . And let  $IC_i[u]$  denote the set of all elements of  $IC_i$  adjacent (as vertices of  $G_M$ ) to at least one country of  $B_u$ . Let  $P_i(u) := \phi_i^{-1}(B_u)$ . **If**  $G_M[B_u]$  is not a cycle graph, **then**  $P_i(N_u) := \phi_i^{-1}(B_u \cup B_v \cup IC_i[u])$  **else**  $P_i(N_u) := \phi_i^{-1}(B_v \cup IC_i[u])$ . **If** there exists a pair  $(p, q)$  of pebbles of  $\phi_i$  such that  $p \neq q, p \in P_i(u), q \in P_i(N_u)$  and the agent subnetwork  $G_A[p \cup q]$  is connected, **then return**  $\{p, q\}$ .
- (②-c) **If**  $G_M[B_u]$  is a cycle graph, **and if** the set  $\phi_i^{-1}(B_u)$  contains a pair of distinct pebbles  $\{p, q\}$  such that the agent subnetwork  $G_A[p \cup q]$  is connected and that the pebble  $p$  can contact the pebble  $q$  along the cycle  $G_M[B_u]$ , **then return**  $\{p, q\}$ .
- (②-d) **If**  $T$  has at least one edge **then set**  $T := T \setminus \{u, {}_uI_v\}$  and **goto** (②-b).

③ **Return**  $\emptyset$ .

**Theorem 13** *Algorithm 5.1 works correctly. Its running time is  $O(|E(G_M)| + (|V(G_M)| + |E(G_A)|)|V(G_A)|)$ .*

**Proof.** We have already explained (just before the description of the algorithm) the correctness of the algorithm.

Now let us estimate the time complexity of the algorithm. Without loss of generality, here we assume that both the graphs  $G_A$  and  $G_M$  are connected. We can achieve the step ② of Algorithm 5.1 in  $O(|E(G_A)| + |E(G_M)|)$ -time by using Theorems 7 and 8. The main body of our algorithm are Algorithm 5.2 and Algorithm 5.3 (i.e. the step ① of Algorithm 5.1). The step ① of Algorithm 5.2 can be done in  $O(|E(G_A)| + |E(G_M)|)$ -time. We keep track of the family  $S(G_i)$  of all the maximal isthmuses of the current board graph  $G_i$  throughout Algorithm 5.2, which is referred to each time when the  $k$ -isthmus trees of  $G_i$  are made at the step ② in Algorithm 5.3. By using the list  $S(G_i)$ , we can keep down the total time of making all the  $k_j$ -isthmus trees  $T_k(G_{i,j})$  ( $j = 1, \dots, l_i$ ) at the step ② in Algorithm 5.3 in  $O(|V(G_M)|)$ -time. The total running time of (②-b) and (②-c) in the step ② of Algorithm 5.3 is  $O(|V(G_M)| + |E(G_A)|)$ . Hence Algorithm 5.3 can be done in  $O(|V(G_M)| +$

$|E(G_A)|$ -time. Because the size of pebbles  $|P_i|$  decreases after the step ⑤ of Algorithm 5.2, the steps ②–⑤ are executed at most  $|V(G_A)| - 1$  times. Hence the total running time of the **ContractiblePair** oracle (i.e. Algorithm 5.3) during Algorithm 5.2 is  $O((|V(G_M)| + |E(G_A)|)|V(G_A)|)$ . The steps ③ and ④ of Algorithm 5.2 can be done in  $O(|V(G_A)|)$ -time. Updating the list  $S(G_i)$  after each step ⑤ of Algorithm 5.2 also can be done in  $O(|V(G_M)|)$ -time. Hence the total running time of the steps ②–⑤ with maintenance of the list  $S(G_i)$  during Algorithm 5.2 is also  $O((|V(G_M)| + |E(G_A)|)|V(G_A)|)$ . Summing all the estimates in the above, the stated overall running time follows.  $\square$

**Theorem 14** *For an ordered pair  $(G_A, G_M)$  of (general) graphs, and for two arrangements  $f, g$  on the pair  $(G_A, G_M)$ , we can construct an explicit sequence of transfers from  $f$  to  $g$  of  $\Theta(|V(G_M)|^2 \cdot |V(G_A)|)$ -length.*

**Proof.** The total number of the **ContractiblePair** oracle calls during Algorithm 5.2, which is the same as the total number of contacts of ‘pebbles’, is clearly at most  $|V(G_A)| - 1$ . Kornhauser et al. [5] proved that the transitivity for the classical pebble motion problem can be done in  $O(|V(G_M)|^2)$  moves and such a sequence of moves can be efficiently generated. By using these facts, we can construct an explicit sequence of transfers from  $f$  to  $g$  of  $O(|V(G_M)|^2 |V(G_A)|)$ -length. On the other hand, Kornhauser et al. [5] also proved that the optimal transformation for the classical pebble motion problem requires  $\Theta(|V(G_M)|^2 \cdot |V(G_A)|)$  moves, which guarantees that our optimal sequence of transfers also requires  $\Theta(|V(G_M)|^2 \cdot |V(G_A)|)$ -length.  $\square$

## 6 Decision of Almightyness

In this section, we prove that the decision of almighty for AAP (and also for SGA) is  $\text{co-}\mathcal{NP}$ -complete.

First we recall the definition of almighty for AAP. Let  $(G_A, G_M)$  be an ordered pair of simple undirected graphs. The pair  $(G_A, G_M)$  is called almighty if the all arrangements on the pair  $(G_A, G_M)$  are t-equivalent each other.



Here let us assume that the input agent network  $G_A$  is a connected graph, because this restriction may be a natural demand from several applications. Unfortunately, we will see that this restriction does not affect the time-complexity of our problem.

**Problem 15 (The decision of almightiness)**

**INSTANCE:** An ordered pair of simple undirected connected graphs  $(G_A, G_M)$ .

**PROBLEM :** Is the ordered pair  $(G_A, G_M)$  almighty?

**Theorem 16** *The decision problem of almightiness for AAP is co- $\mathcal{NP}$ -complete.*

For example, suppose that  $C$  is a circuit graph,  $G_M$  is a graph whose complement  $\overline{G_M}$  has a Hamiltonian circuit, and  $|V(C)| = |V(G_M)|$  holds. Then there is an arrangement  $f : V(C) \rightarrow V(G_M)$  of the ordered graph pair  $(C, G_M)$  such that the  $f$  is bijective and that every edge  $\{u, v\} (\in E(C))$  is transformed into a non-edge  $\{f(u), f(v)\} (\notin E(G_M))$ . And hence in this case, the pair  $(C, G_M)$  is not almighty. The Hamiltonian circuit problem (HC, for short) is a well known  $\mathcal{NP}$ -complete problem. However we must not run away with idea that this example directly shows the co- $\mathcal{NP}$ -completeness of our problem, simply because the almightiness is so restrictive condition that, even if the complement of  $G_M$  does not have a Hamiltonian circuit, there exist many types of candidate of such  $G_M$  that the pair  $(C, G_M)$  is not almighty. In stead of analyzing and classifying the complicated variety of the shapes of  $G_M$  for which the pair  $(C, G_M)$  is not almighty, let us take a strategy to provide appropriate gadgets for and to attach them to each of the circuit graph  $C$  and the input graph  $G_M$ .

Before prove Theorem 16, let us briefly analyze the time-complexity of the following restricted version of HC.

**Problem 17 (A restricted version of HC)**

**INSTANCE:** A simple undirected connected graph  $H$  whose complement  $\overline{H}$  is also connected.

**PROBLEM :** Is the graph  $H$  has a Hamiltonian circuit?

**Lemma 18** *The Problem 17 is  $\mathcal{NP}$ -complete.*

**Proof of Lemma 18.** It is clear that Problem 17 is in  $\mathcal{NP}$ . Now we will show a polynomial-time reduction from HC to Problem 17. Without loss of generality, we assume that the input graph  $G$  of HC has at least 3 vertices. Let  $v$  be an arbitrary vertex of  $G$ , and let  $\{a, b, c\}$  be three vertices outside of  $V(G)$ . Let us construct a new graph  $H$  by combining  $G$  and the three vertices  $\{a, b, c\}$  as follows:

$$\begin{aligned} V(H) &:= V(G) \cup \{a, b, c\}; \\ E(H) &:= E(G) \cup \{\{v, a\}, \{a, b\}, \{b, c\}\} \cup \{\{c, x\} : \{x, v\} \in E(G)\}. \end{aligned}$$

Then it is easy to see that the graph  $H$  has a Hamiltonian circuit if and only if the original graph  $G$  has a Hamiltonian circuit. Furthermore, the complement  $\overline{H}$  of the graph  $H$  is connected. Clearly, this reduction can be done in  $O(|V(G)|)$ -time.  $\square$

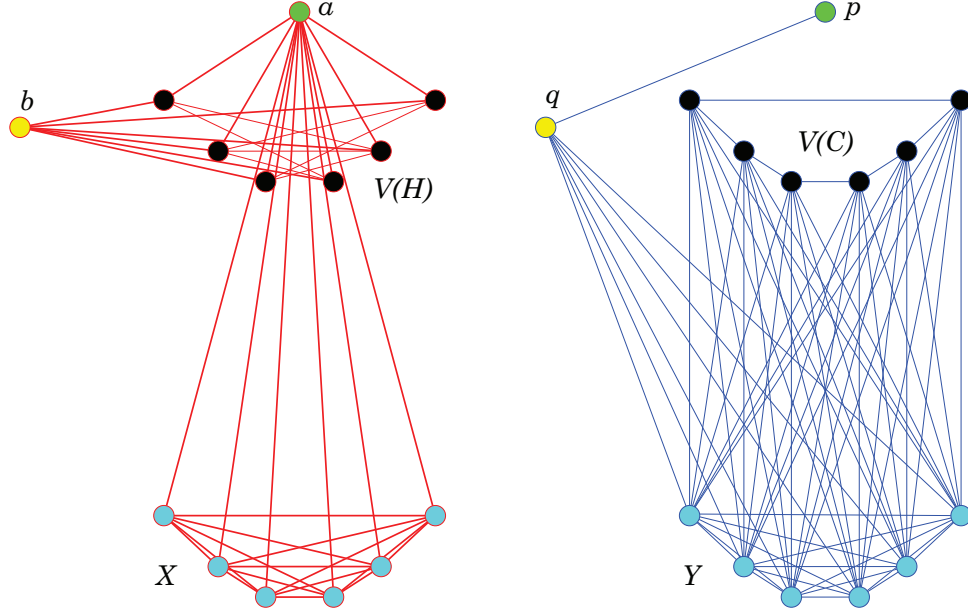
**Proof of Theorem 16.** If an ordered pair  $(G_A, G_M)$  of graphs is not almighty, as evidence for this fact, we can bring forward two arrangements  $f$  and  $g$  of the pair  $(G_A, G_M)$  such that  $f \not\equiv g$  holds. Moreover, as we have shown in the section 5, we can verify the correctness of this evidence in polynomial-time. Hence Problem 15 is in  $\text{co-}\mathcal{NP}$ .

Next, in order to prove the  $\text{co-}\mathcal{NP}$  completeness of Problem 15, we will reduce the complement of Problem 17 to Problem 15 in polynomial-time. Let  $H$  be an input graph of Problem 17,  $n := |V(H)|$  be its order. Let  $X$  be a set of  $n$  vertices outside of  $V(H)$ , and let  $\{a, b\}$  be a pair of distinct vertices outside of  $V(H) \cup X$ . By using these, we define the following new graph  $G_A$ :

$$\begin{aligned} V(G_A) &:= V(H) \cup X \cup \{a, b\}; \\ E(G_A) &:= E(\overline{H}) \cup \binom{X}{2} \cup \{\{a, u\} : u \in V(H) \cup X\} \cup \{\{b, h\} : h \in V(H)\}. \end{aligned}$$

Then, let  $C$  be a cycle graph such that its vertex-set is  $V(C) := \{c_1, \dots, c_n\}$  and that its edge-set is  $E(C) := \{\{c_1, c_2\}, \dots, \{c_{n-1}, c_n\}, \{c_n, c_1\}\}$ . Let  $Y$  be a set of  $n$  vertices outside of  $V(C)$ , and let  $\{p, q\}$  be a pair of distinct vertices outside of  $V(C) \cup Y$ . By using these, we define the following new graph  $G_M$ :

$$\begin{aligned} V(G_M) &:= V(C) \cup Y \cup \{p, q\}; \\ E(G_M) &:= E(C) \cup \binom{Y}{2} \cup (Y \times (V(C) \cup \{q\})) \cup \{\{p, q\}\}. \end{aligned}$$



**Fig.3.** The two graphs  $G_A$  and  $G_M$  in the proof of Theorem 16.

Now suppose that  $H$  has a Hamiltonian circuit. Then there exists a bijection  $f : V(G_A) \rightarrow V(G_M)$  such that  $f(a) = p$ ,  $f(b) = q$ ,  $f(X) = Y$ ,  $f(V(H)) = V(C)$  and, for every edge  $\{u, u'\}$  of the complement  $\overline{H}$  of the graph  $H$ , the pair  $\{f(u), f(u')\}$  is a non-edge of  $C$  (and hence of  $G_M$ ). This bijection  $f$  can be thought as an arrangement of the pair  $(G_A, G_M)$ , and it is easy to see that this  $f$  cannot be t-equivalent to any arrangement  $g$  such that  $\exists v \in V(H) \cup \{a, b\}, g(v) \neq f(v)$ . And hence the pair  $(G_A, G_M)$  is not almighty.

On the contrary, let us assume that  $H$  does not have any Hamiltonian circuit. We will show the fact that every arrangement  $f$  of the pair  $(G_A, G_M)$  is t-equivalent to the special arrangement  $g : V(G_A) \rightarrow \{p\}$ , which proves that the pair  $(G_A, G_M)$  is almighty.

First we prove the following technical lemma.

**Lemma 19** *For every vertex  $v$  in  $V(G_M) \setminus \{f(a)\}$ , either  $v \notin f(V(H) \cup \{b\})$*

or  $v \notin f(X)$  holds.

**Proof of Lemma 19.** The graph  $G_A[f^{-1}(v)]$  must be connected and every connected subgraph of  $G_A - a$  is a subgraph of either  $G_A[\{b\} \cup V(H)]$  or  $G_A[X]$ , which proves the lemma.  $\square$

We divide our proof into the several cases.

**Case 1**  $f(a) \in Y$ .

**Subcase 1.1** There exists a vertex  $h$  in  $V(H)$  such that  $f(h) \in V(G_M) \setminus \{p\}$ .

Because the pair  $\{a, h\}$  is an edge of  $G_A$  and the pair  $\{f(a), f(h)\}$  is an edge of  $G_M$ , we may transfer the graph  $G_A[f^{-1}(f(h))]$  onto the vertex  $f(a)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{a, h\}))]$ . Next, because every vertex of  $G_A$  is adjacent to at least one of  $\{a, h\}$ , we can move all the vertices of  $f^{-1}(V(G_M) \setminus \{p\})$  onto the vertex  $f(a)$  of  $G_M$ . Then,  $G_A[f^{-1}(V(G_M) \setminus \{p\})]$  is on the vertex  $f(a)$  of  $G_M$  and all the other vertices of  $G_A$  are on the vertex  $p$  of  $G_M$ . Finally, we may transfer the graph  $G_A[f^{-1}(V(G_M) \setminus \{p\})]$  onto the vertex  $p$  and merge it into  $G_A$  on  $p$ . Hence the arrangement  $f$  of the pair  $(G_A, G_M)$  is t-equivalent to the special arrangement  $g : V(G_A) \rightarrow \{p\}$ .

**Subcase 1.2**  $f(V(H)) = \{p\}, f(b) = q$ .

Since every vertex of  $V(H)$  is adjacent to  $b$  and the pair  $\{p, q\}$  is an edge of  $G_M$ , we can transfer the graph  $G_A[f^{-1}(p)]$  onto the vertex  $q$  of  $G_M$  and merge it into  $G_A[f^{-1}(\{p, q\})]$ , which means that the case is reduced to **Subcase 1.1**.

**Subcase 1.3**  $f(V(H)) = \{p\}, f(b) \neq q$ .

Since every vertex in  $f^{-1}(q)$  is adjacent to the vertex  $a$  and the pair  $\{q, f(a)\}$  is an edge of  $G_M$ , we may transfer the graph  $G_A[f^{-1}(q)]$  onto the vertex  $f(a)$  of  $G_M$  and merge it into  $G_A[f^{-1}(\{q, f(a)\})]$ . Then the vertex  $q$  will be unoccupied, and hence we can transfer the graph  $G_A[f^{-1}(p)]$  onto the unoccupied vertex  $q$ , which means that the case is reduced to **Subcase 1.1**.

**Case 2**  $f(a) \in V(C)$ .

Note that Lemma 19 tells us that  $p \notin f(V(H)) \cap f(X)$  holds.

**Subcase 2.1** There exists a vertex  $u$  in  $V(H) \cup X$  such that  $f(u) \in Y$ .

In this case, since the pair  $\{a, u\}$  is an edge of  $G_A$  and the pair  $\{f(a), f(u)\}$  is an edge of  $G_M$ , we can transfer the graph  $G_A[f^{-1}(f(a))]$  onto the vertex  $f(u)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{a, u\}))]$ , which means that the case is reduced to **Case 1**.

**Subcase 2.2**  $f(V(H) \cup X) \cap Y = \emptyset$  and there exist a vertex  $w$  in  $V(C) \cup \{q\}$  such that  $|f^{-1}(w)| \geq 2$ .

In this case, we can regard this connected subgraph  $G_A[f^{-1}(w)]$  as if a single pebble on the board graph  $G_M$ . Because  $b$  is a unique vertex of  $G_A$  which is not adjacent to  $a$ , at least one vertex  $y$  of  $Y$  is unoccupied. We can move the pebble  $G_A[f^{-1}(w)]$  onto this unoccupied vertex  $y$ , and hence the case is reduced to either **Case 1** or **Subcase 2.1**.

**Subcase 2.3**  $f(V(H) \cup X) \cap Y = \emptyset$  and every vertex  $w$  in  $V(C) \cup \{q\}$  satisfies  $|f^{-1}(w)| \leq 1$ .

In this case, because  $p \notin f(V(H) \cup \{b\}) \cap f(X)$ , the Pigeonhole principle shows that the set  $X \cup V(H)$  includes a vertex  $u$  such that  $\{f(a), f(u)\}$  is an edge of the cycle graph  $C$ . Since  $\{a, u\}$  is an edge of  $G_A$ , we can merge  $a$  and  $u$  into the graph  $G_A[\{a, u\}]$  and put it on the vertex  $f(a)$  of  $G_M$ . Moreover, we can regard this  $G_A[\{a, u\}]$  as if a single pebble on the board graph  $G_M$ . Since at least one vertex  $y$  of  $Y$  is unoccupied, we can move this pebble  $G_A[\{a, u\}]$  onto the unoccupied vertex  $y$  of  $G_M$ , which means that the case is reduced to **Case 1**.

**Case 3**  $f(a) = q$ .

**Subcase 3.1** The set  $V(H) \cup X$  includes a vertex  $u$  such that  $f(u) \in Y$ .

The pair  $\{a, u\}$  is an edge of  $G_A$  and the pair  $\{f(a), f(u)\}$  is an edge of  $G_M$ . Hence we can transfer the graph  $G_A[f^{-1}(f(a))]$  onto the vertex  $f(u)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{a, u\}))]$ , which means that the case is reduced to **Case 1**.

**Subcase 3.2**  $f(V(H) \cup X) \cap Y = \emptyset$  and the set  $V(H) \cup X$  includes a vertex  $u$  such that  $f(u) = p$ .

In the same way as the previous case, the pair  $\{a, u\}$  is an edge of  $G_A$  and the pair  $\{f(a), f(u)\}$  is an edge of  $G_M$ . Hence we can transfer the graph  $G_A[f^{-1}(f(u))]$  onto the vertex  $q$  of  $G_M$  and merge it into  $G_A[f^{-1}(\{p, q\})]$ . Because  $b$  is a unique vertex of  $G_A$  which is not adjacent to  $a$ , at least one vertex  $y$  of  $Y$  is unoccupied. We can transfer the graph  $G_A[f^{-1}(\{p, q\})]$  onto this unoccupied vertex  $y$  of  $G_M$ , and hence the case is reduced to **Case 1**.

**Case 4**  $f(a) = p$ .

**Subcase 4.1**  $f(b) = p$ .

Since every vertex of  $G_A$  is adjacent to at least one of the pair  $\{a, b\}$ , we can transfer  $G_A[f^{-1}(p)]$  onto the vertex  $q$  of  $G_M$  and merge it into  $G_A[f^{-1}(\{p, q\})]$ , which means that the case is reduced to **Case 3**.

**Subcase 4.2** There exists a vertex  $u$  in  $V(H) \cup X$  such that  $f(u) = q$ .

The pair  $\{a, u\}$  is an edge of  $G_A$  and the pair  $\{f(a), f(u)\}$  is an edge of  $G_M$ . Hence we can transfer the graph  $G_A[f^{-1}(f(a))]$  onto the vertex  $q$  of  $G_M$  and merge it into  $G_A[f^{-1}(\{p, q\})]$ , which means that the case is reduced to **Case 3**.

**Subcase 4.3**  $q \notin f(V(H) \cup X)$  and  $f(b) \neq p$  and there exists a vertex  $h$  of  $V(H)$  such that  $f(h) \in Y$ .

Because  $f(h) \in Y$  is adjacent to every vertex of  $V(C) \cup Y \cup \{q\}$  and  $\{b, h\}$  is an edge of  $G_A$ , we can transfer the graph  $G_A[f^{-1}(f(b))]$  onto the vertex  $f(h)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{b, h\}))]$ . Moreover we can also transfer the graph  $G_A[f^{-1}(f(\{b, h\}))]$  onto the vertex  $q$  of  $G_M$  because either  $f^{-1}(q) = \emptyset$  or  $f(b) = q$  holds. Hence the case is reduced to **Subcase 4.2**.

**Subcase 4.4**  $q \notin f(V(H) \cup X)$  and  $f(b) \in Y$  and  $f(V(H)) \subseteq V(C)$ .

For every vertex  $h$  in  $V(H)$ , the pair  $\{h, b\}$  is an edge of  $G_A$  and the pair  $\{f(h), f(b)\}$  is an edge of  $G_M$ , and hence we can transfer the graph  $G_A[f^{-1}(f(h))]$  onto the vertex  $f(b)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{b, h\}))]$ , which means that the case is reduced to **Subcase 4.3**.

**Subcase 4.5**  $q \notin f(V(H) \cup X)$  and  $f(b) \notin Y$  and  $f(V(H)) \subseteq V(C)$ .

In this case, we have that  $f^{-1}(Y) \subseteq X$ , and hence, for every vertex  $x$  in  $f^{-1}(Y)$ , we may transfer the graph  $G_A[f^{-1}(Y)]$  onto the vertex  $f(x)$  of  $G_M$ . Because  $|Y| = n \geq 2$ ,  $Y$  includes at least one vertex  $y$  other than  $f(x)$ . We may assume that this vertex  $y$  will be unoccupied. Because  $f(V(H)) \subseteq V(C)$ , and because  $H$  does not have any Hamiltonian circuit,  $V(H)$  includes a pair of distinct vertices  $\{h_1, h_2\}$  such that either the pair  $\{f(h_1), f(h_2)\}$  is an edge of  $C$  or  $f(h_1) = f(h_2)$  holds. Hence we may transfer the graph  $G_A[f^{-1}(f(h_1))]$  onto the vertex  $f(h_1)$  of  $G_M$  and merge it into  $G_A[f^{-1}(f(\{h_1, h_2\}))]$ . Next, we may transfer the graph  $G_A[f^{-1}(f(\{h_1, h_2\}))]$  onto the unoccupied vertex  $y$  of  $G_M$ , which means that the case is reduced to **Subcase 4.3**.

Now our proof is completed. □

**Corollary 20** *The decision problem of almightiness for SGA is co- $\mathcal{NP}$ -complete.*

Its proof is almost the same as the one of Theorem 16 and will be omitted. (Please use the one vertex  $v_X$  (resp.  $v_Y$ ) except for the gadget  $X$  (resp.  $Y$ ).)

## References

- [1] A. Archer, A modern treatment of the 15 puzzle, *American Math. Monthly* **106** (1999), 793–799.
- [2] V. Auletta, A. Monti, M. Parente, and P. Persiano, A linear-time algorithm for the feasibility of pebble motion in trees, *Algorithmica* **23** (1999), 223–245.
- [3] G. Calinescu, A. Dumitrescu, and J. Pach, Reconfigurations in Graphs and Grids, *Proceedings of the 7-th Latin American Symposium on Theoretical Informatics*, LATIN 2006, LNCS 3887, 262–273.
- [4] W. W. Johnson, Notes on the 15 puzzle. I., *American J. Math.* **2** (1879), 397–399.

- [5] D. Kornhauser, G. Miller, and P. Spirakis, Coordinating pebble motion on graphs, the diameter of permutation groups, and applications, *Proceedings of the 25-th Symposium on Foundations of Computer Science*, (FOCS '84), 241–250.
- [6] C. Papadimitriou, P. Raghavan, M. Sudan, and H. Tamaki, Motion planning on a graph, *Proceedings of the 35-th Symposium on Foundations of Computer Science*, (FOCS '94), 511–520.
- [7] D. Ratner and M. Warmuth, Finding a shortest solution for the  $(N \times N)$ -extension of the 15-puzzle is intractable, *J. Symbolic Computation* **10** (1990), 111–137.
- [8] W. E. Story, Notes on the 15 puzzle. II., *American J. Math.* **2** (1879), 399–404.
- [9] R. M. Wilson, Graph puzzles, homotopy, and the alternating group, *J. Combin. Theory Ser. B* **16** (1974), 86–96.